J. Math. Anal. Appl. 392 (2012) 136–147

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Linear stability analysis of symmetric periodic simultaneous binary collision orbits in the planar pairwise symmetric four-body problem

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ARTICLE INFO

Article history:
Received 5 December 2011
Available online 11 April 2012
Submitted by Willy Sarlet

Keywords:
N-body problem
Singular periodic orbits
Linear stability

ABSTRACT

We apply the symmetry reduction method of Roberts to numerically analyze the linear stability of a one-parameter family of symmetric periodic orbits with regularizable simultaneous binary collisions in the planar pairwise symmetric four-body problem with a mass $m \in (0, 1]$ as the parameter. This reduces the linear stability analysis to the computation of two eigenvalues of a 3×3 matrix for each $m \in (0, 1]$ obtained from numerical integration of the linearized regularized equations along only the first one-eighth of each regularized periodic orbit. The results are that the family of symmetric periodic orbits with regularizable simultaneous binary collisions changes its linear stability type several times as m varies over $(0, 1]$, with linear instability for m close or equal to 0.01, and linear stability for m close or equal to 1.

1. Introduction

In Hamiltonian systems like the Newtonian N-body problem, linear stability of a periodic orbit is necessary but insufficient for its nonlinear stability [1]. When the periodic orbit is not a relative equilibrium, the characteristic multipliers are typically found by computing its monodromy matrix, i.e., by numerically integrating the linearized equations along the periodic orbit over a full period (in which the periodic orbit and its period are typically computed numerically as well). For a symmetric periodic orbit, Roberts [2] developed a symmetry reduction method by which the nontrivial characteristic multipliers are computed by numerical integration of the linearized equations along the periodic orbit over a fraction of the full period. He applied this symmetry reduction method to show that numerically the Montgomery–Chenciner figure-eight periodic orbit with equal masses [3] is linearly stable [2]; the numerical integration of the linearized equations along the periodic orbit only needed to go over one-twelfth of the full period.

We apply Roberts' symmetry reduction method to a one-parameter family of symmetric singular periodic orbits in the planar pairwise symmetric four-body problem (PPS4BP) where the parameter is a mass $m \in (0, 1]$ and the singularities are regularizable simultaneous binary collisions (SBCs). We recall in Section 2 the notation we used in [4] for the PPS4BP. (The PPS4BP is the Caledonian symmetric four-body problem [5] without its collinear restrictions on the initial conditions.) To compute the nontrivial characteristic multipliers of these periodic orbits we numerically integrated the linearized regularized equations along each regularized periodic orbit over only one-eighth of its period. This shows that numerically these symmetric singular periodic orbits experience several changes in their linear stability type (linearly stable, spectrally stable, or linearly unstable) as m is varied over $(0, 1]$.

This is a correction of our previous numerical investigations of the linear stability of these symmetric singular periodic orbits. We numerically computed [4] the monodromy matrix and its eigenvalues for each regularized symmetric periodic orbit starting at $m = 1.00$ and decreasing by 0.01 until $m = 0.01$. This seemed to indicate that the periodic orbits were

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doi:10.1016/j.jmaa.2012.03.022

linearly stable for m in the interval [0.54, 1.00] and linearly unstable for m in the interval [0.01, 0.53]. This agreed with the stability and instability suggested by our long-term numerical integrations of the regularized equations starting at a numerically computed approximation of each periodic orbit's initial conditions. However, our numerical estimates of the monodromy matrices failed to accurately account for the trivial characteristic multiplier 1 of algebraic multiplicity 4: instead of getting 1 as an eigenvalue for each monodromy matrix, we were getting two pairs of eigenvalues, one pair of positive eigenvalues with one larger than 1 and the other smaller than 1, and one pair of complex conjugates close to 1. As m passed below 0.61, the pair of real eigenvalues began to move away from 1, so much so, that below $m = 0.21$, we had a real positive eigenvalue of the monodromy matrix whose value exceeded the limits of MATLAB. This calls into question the conclusions of our first attempt at determining for what values of m the symmetric singular periodic orbits in the PPS4BP were linearly stable and linearly unstable.

We thus proceeded to use Roberts' symmetry reduction method because it factors out, in an analytic manner, two of the trivial characteristic multipliers, leaving the numerical computations to estimate the two pairs of nontrivial characteristic multipliers and one pair of trivial characteristic multipliers. The details of these computations are given in Section 4. Two surprises here are two nonintersecting open intervals, one containing [0.21, 0.22] and the other containing [0.23, 0.26], where we have linear stability. Our long-term numerical integrations of the regularized equations for these periodic orbits (starting at our numerical approximations of their initial conditions and over 100 932 periods) suggested instability for m in these two intervals. We further refined the numerical computations for m between 0.19 and 0.21, between 0.22 and 0.23, between 0.26 and 0.27, between 0.53 and 0.54, and between 0.54 and 0.55 by increments of 0.001 to get better estimates of all the values of m where the linear stability type changes. This gives six critical values m_i, $i = 2, 3, 4, 5, 6, 7$, where

$$0.19 < m_2 < m_3 < m_4 < m_5 < m_6 < m_7 < 0.55$$

at which the periodic orbits are spectrally stable, with linearly stability for m in the intervals

$$(m_2, m_3),\ (m_3, m_4),\ (m_4, m_5),\ (m_6, m_7),\ (m_7, 1]$$

and linear instability for m in the intervals

$$(0, m_2),\ (m_5, m_6).$$

There is a value m_1 in (0.09, 0.10) at which two of the nontrivial characteristic multipliers are -1, but this does not change the linear stability type of the periodic orbit for m_1 as the other nontrivial characteristic multipliers give linear instability at m_1.

Such changes in the linear stability type of mass-parameterized families of symmetric periodic orbits with regularizable collisions have been found in other N-body problems. The Schubart orbit in the collinear three-body problem [6–10] has the inner body alternating between binary collisions with the two outer bodies. These are linearly stable for certain choices of the three masses [11]. Linearly stable non-Schubart orbits have also been found in the collinear three-body problem for certain choices of the masses [12–14]. The Schubart-like orbit in the collinear symmetric four-body problem [15–18,10], alternates between a binary collision of the two inner bodies and a SBC of the two outer pairs of bodies. If the masses in the collinear symmetric four-body problem are, from left to right, 1, m, m, and 1, then linear stability occurs when $0 < m < 2.83$ and $m > 35.4$ with linear instability for $2.83 < m < 35.4$ by numerical computation of their linear stability indices [16] (a method which requires numerical integration of the regularized equations over a full period) and corroborated by Roberts' symmetry reduction method [19]. The symmetric singular periodic orbit in the fully symmetric planar four-body equal mass problem [20,10] (in which the position of one of the bodies determines the positions of the remaining three bodies) alternates between distinct SBCs and has been shown to be linearly stable, with respect to symmetrically constrained linear perturbations, by Roberts' symmetry reduction method [19]. The linearly stable symmetric singular periodic SBC orbit with $m = 1$ in the PPS4BP is the analytic extension [4] of the linearly stable symmetric singular periodic orbit in the fully symmetric planar four-body equal mass problem [19].

2. The PPS4BP

We recall from [4] the relevant notations for the PPS4BP, its regularized Hamiltonian, and properties of the regularized one-parameter family of symmetric SBC period orbits. In the PPS4BP the positions of the four planar bodies are

$$(x_1, x_2),\ (x_3, x_4),\ (-x_1, -x_2),\ (-x_3, -x_4),$$

where the corresponding masses are 1, m, 1, m with $0 < m \leq 1$. With t as the time variable and $\dot{} = d/dt$, the momenta for the four bodies are

$$(\omega_1, \omega_2) = 2(\dot{x}_1, \dot{x}_2), (\omega_3, \omega_4) = 2m(\dot{x}_3, \dot{x}_4), -(\omega_1, \omega_2), -(\omega_3, \omega_4).$$

The Hamiltonian for the PPS4BP is

$$H = \frac{1}{4}\left[\omega_1^2 + \omega_2^2\right] + \frac{1}{4m}\left[\omega_3^2 + \omega_4^2\right] - \frac{1}{2\sqrt{x_1^2 + x_2^2}} - \frac{2m}{\sqrt{(x_3 - x_1)^2 + (x_4 - x_2)^2}}$$
$$- \frac{2m}{\sqrt{(x_1 + x_3)^2 + (x_2 + x_4)^2}} - \frac{m^2}{2\sqrt{x_3^2 + x_4^2}}.$$

The angular momentum for the PPS4BP is

$$A = x_1\omega_2 - x_2\omega_1 + x_3\omega_4 - x_4\omega_3.$$

A regularizable simultaneous binary collision occurs when $x_3 = x_1 \neq 0$ and $x_4 = x_2 \neq 0$ (in the first and third quadrants), and also when $x_3 = -x_1 \neq 0$ and $x_4 = -x_2 \neq 0$ (in the second and fourth quadrants). Initial conditions for the symmetric SBC periodic orbits in the PPS4BP when $m = 1$ are given in [4], and when $m = 0.539$ they are

$$x_1 = 2.11421, \qquad x_2 = 0, \qquad x_3 = 0, \qquad x_4 = 1.01146,$$
$$\omega_1 = 0, \qquad \omega_2 = 0.18151, \qquad \omega_3 = 0.70392, \qquad \omega_4 = 0.$$

For other choices of m in $(0, 1)$, the initial conditions for the symmetric SBC periodic orbits in the PPS4BP can be found in [4].

2.1. The regularized Hamiltonian

We define new variables $u_1, u_2, u_3, u_4, v_1, v_2, v_3$, and v_4 related to the variables $x_1, x_2, x_3, x_4, \omega_1, \omega_2, \omega_3$, and ω_4 by the canonical transformation

$$\begin{aligned}
x_1 &= (1/2)(u_1^2 - u_2^2 + u_3^2 - u_4^2)\\
x_2 &= u_1u_2 + u_3u_4,\\
x_3 &= (1/2)(u_3^2 - u_4^2 - u_1^2 + u_2^2),\\
x_4 &= u_3u_4 - u_1u_2,\\
\omega_1 &= \frac{v_1u_1 - v_2u_2 + v_1u_2 + v_1u_2 + v_2u_1}{2(u_1^2 + u_2^2)},\\
\omega_2 &= \frac{v_3u_3 - v_4u_4 + v_3u_4 + v_4u_3}{2(u_3^2 + u_4^2)},\\
\omega_3 &= \frac{-v_1u_1 + v_2u_2 + v_1u_2 + v_2u_1}{2(u_1^2 + u_2^2)},\\
\omega_4 &= \frac{-v_3u_3 + v_4u_4 + v_3u_4 + v_4u_3}{2(u_3^2 + u_4^2)}.
\end{aligned}$$

In extended phase space, the variables are $u_1, u_2, u_3, u_4, \hat{E}, v_1, v_2, v_3, v_4$, and t, where $\hat{E}$ is the energy. If we set

$$\begin{aligned}
&M_1 = v_1u_1 - v_2u_2, \qquad M_2 = v_1u_2 + v_2u_1,\\
&M_3 = v_3u_3 - v_4u_4, \qquad M_4 = v_3u_4 + v_4u_3,\\
&M_5 = u_1^2 - u_2^2 + u_3^2 - u_4^2, \qquad M_6 = 2u_1u_2 + 2u_3u_4,\\
&M_7 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \qquad M_8 = 2u_1u_2 - 2u_3u_4,
\end{aligned}$$

then the regularized Hamiltonian for the PPS4BP in extended phase space is

$$\begin{aligned}
\hat{\Gamma} = \frac{dt}{ds}(H - \hat{E}) &= \frac{1}{16}\left(1 + \frac{1}{m}\right)\left((v_1^2 + v_2^2)(u_3^2 + u_4^2) + (v_3^2 + v_4^2)(u_1^2 + u_2^2)\right) + \frac{1}{8}\left(1 - \frac{1}{m}\right)(M_3M_1 + M_4M_2)\\
&\quad - \frac{(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{\sqrt{M_5^2 + M_6^2}} - 2m(u_1^2 + u_2^2 + u_3^2 + u_4^2) - \frac{m^2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{\sqrt{M_7^2 + M_8^2}} - \hat{E}(u_1^2 + u_2^2)(u_3^2 + u_4^2),
\end{aligned}$$

where

$$\frac{dt}{ds} = (u_1^2 + u_2^2)(u_3^2 + u_4^2)$$

is the regularizing change of time for this Levi-Civita regularization. The angular momentum in the new variables is

$$A = \frac{1}{2}\left[-v_1u_2 + v_2u_1 - v_3u_4 + v_4u_3\right].$$

Let $' = d/ds$,

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

for I the 4×4 identity matrix, and ∇ is the gradient with respect to the variables

$$z = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4).$$

The regularized Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ is

$$z' = J\nabla\hat{\Gamma}(z). \tag{1}$$

The energy $\hat{E}$ is conserved because

$$\hat{E}' = \frac{\partial\hat{\Gamma}}{\partial t} = 0.$$

2.2. The symmetric periodic SBC orbits in the regularized PPS4BP

For $m = 1$, we have analytically proven [4] the existence and symmetries of a symmetric periodic SBC orbit $\gamma(s;\,1)$, with period $T = 2\pi$, $\hat{E} \approx -2.818584789$, and $A = 0$ for the regularized PPS4BP on the level set $\hat{\Gamma} = 0$. The initial conditions of $\gamma(s;\,1)$ at $s = 0$ satisfy

$$u_3(0;\,1) = u_1(0;\,1), \qquad u_4(0;\,1) = -u_2(0;\,1),$$
$$v_3(0;\,1) = -v_1(0;\,1), \qquad v_4(0;\,1) = v_2(0;\,1).$$

The symmetries of $\gamma(s;\,1)$ are $S_F\gamma(s;\,1) = \gamma(s + \pi/2;\,1)$ and $S_G\gamma(s;\,1) = \gamma(\pi/2 - s;\,1)$ where

$$S_F = \begin{bmatrix} 0 & F & 0 & 0 \\ -F & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & -F & 0 \end{bmatrix}, \qquad S_G = \begin{bmatrix} -G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & -G \end{bmatrix},$$

for

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and 0 in S_F and S_G is the 2×2 zero matrix. Using a scaling of periodic orbits in the PPS4BP, we [4] numerically continued the symmetric SBC periodic orbit $\gamma(s;\,1)$ to symmetric periodic SBC orbits $\gamma(s;\,m)$ with $A = 0$ for $0 < m < 1$ at 0.01 decrements with fixed period $T = 2\pi$ and varying energies $\hat{E}(m)$ using trigonometric polynomial approximations that ensured the symmetries $S_F\gamma(s;\,m) = \gamma(s + \pi/2;\,m)$ and $S_G\gamma(s;\,m) = \gamma(\pi/2 - s;\,m)$. For all $0 < m \leq 1$, the components of $\gamma(0;\,m)$ satisfy

$$u_3(0;\,m) = u_1(0;\,m), \qquad u_4(0;\,m) = -u_2(0;\,m), \tag{2}$$
$$v_3(0;\,m) = -v_1(0;\,m), \qquad v_4(0;\,m) = v_2(0;\,m). \tag{3}$$

For all $0 < m \leq 1$, regularized SBCs occur at $s = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, where at the first and third times we have $v_3^2 + v_4^2 = 0$ while at the second and fourth times we have $v_1^2 + v_2^2 = 0$. The regularized symmetric periodic orbit $\gamma(s;\,m)$, in going from $s = 0$ to $s = 2\pi$, corresponds in the original Hamiltonian system in the physical plane to two full periods of oscillation of a symmetric singular periodic orbit, whose only singularities are regularizable SBCs.

Each regularized symmetric periodic orbit $\gamma(s;\,m)$ has the trivial characteristic multiplier 1 of algebraic multiplicity at least 4. This is because the regularized Hamiltonian $\hat{\Gamma}$ and the angular momentum A are first integrals for the regularized Hamiltonian system (1), and because of the time translation along the periodic orbits and SO(2) rotations of the periodic orbits (see [1]).

3. Linear stability of periodic SBC orbits

We apply Roberts' symmetry reduction method [2] to the one-parameter family of periodic orbits $\gamma(s;\,m)$, $0 < m \leq 1$, of fixed period 2π, in the regularized Hamiltonian system (1). Let $\nabla^2\hat{\Gamma}$ denote the symmetric matrix of second-order partials of $\hat{\Gamma}$ with respect to the components of $z = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$. It is easily shown, that if $Y(t)$ is the fundamental matrix solution of the linearized equations along $\gamma(s;\,m)$,

$$\xi' = J\nabla^2\hat{\Gamma}(\gamma(s))\xi, \quad \xi(0) = Y_0,$$

for an invertible Y_0, then the eigenvalues of $Y_0^{-1}Y(2\pi)$ are indeed the characteristic multipliers of $\gamma(s;\,m)$.

3.1. Stability reductions using symmetries

We use the symmetries of $\gamma(s;\,m)$ to show that $Y_0^{-1}Y(2\pi)$ can be factored in part by terms of the form $Y(\pi/4)$, that is, one-eighth of the period of $\gamma(s;\,m)$. Thus the symmetries of $\gamma(s;\,m)$ will reduce the analysis of its linear stability type to the numerical computation of $Y(\pi/4)$.

Lemma 3.1. *For each $0 < m \leq 1$, there exists a matrix W such that $Y_0^{-1}Y(2\pi) = W^4$ where $W = \Lambda D$ for involutions Λ and D with $\Lambda = Y_0^{-1}S_F^TS_GY_0$ and $D = B^{-1}S_GB$ for $B = Y(\pi/4)$.*

Proof. Each $\gamma(s;m)$ satisfies $S_F\gamma(s;m) = \gamma(s+\pi/2;m)$. Then (by [2], see also [19]), we have that

$$Y(k\pi/2) = S_F^k Y_0 (Y_0^{-1} S_F^T Y(\pi/2))^k$$

holds for all $k \in \mathbb{N}$. Since $S_F^4 = I$, taking $k = 4$ gives

$$Y(2\pi) = Y_0(Y_0^{-1}S_F^T Y(\pi/2))^4. \tag{4}$$

Furthermore, each $\gamma(s;m)$ satisfies $S_G\gamma(s;m) = \gamma(\pi/2 - s;m)$. Then (by [2], see also [19]), for

$$B = Y(\pi/4)$$

we have that

$$Y(\pi/2) = S_G Y_0 B^{-1} S_G^T B = S_G Y_0 B^{-1} S_G B, \tag{5}$$

where we have used $S_G^T = S_G$. Combining equations (4) and (5) gives the factorization

$$Y(2\pi) = Y_0(Y_0^{-1}S_F^T S_G Y_0 B^{-1} S_G B)^4.$$

By setting

$$Q = S_F^T S_G \quad \text{and} \quad W = Y_0^{-1} Q Y_0 B^{-1} S_G B,$$

we obtain

$$Y_0^{-1} Y(2\pi) = (Y_0^{-1} Q Y_0 B^{-1} S_G B)^4 = W^4,$$

where

$$\Lambda = Y_0^{-1} Q Y_0 \quad \text{and} \quad D = B^{-1} S_G B$$

are both involutions, i.e., $\Lambda^2 = D^2 = I$. □

3.2. A choice of Y_0

The matrix $Q = S_F^T S_G$ that appears in Λ is orthogonal since S_F and S_G are both orthogonal. Furthermore, Q is symmetric and its eigenvalues are ± 1, each of multiplicity 4. An orthogonal basis for the eigenspace $\ker(Q - I)$ is

$$\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\0\\0\\0\\0\end{bmatrix},$$

and an orthogonal basis for the eigenspace $\ker(Q + I)$ is

$$\begin{bmatrix}0\\0\\0\\0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}.$$

We look for an appropriate choice of Y_0 such that

$$\Lambda = Y_0^{-1} Q Y_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{6}$$

Lemma 3.2. *There exists an orthogonal and symplectic Y_0 such that Eq. (6) holds.*

Proof. Since the components of $\gamma(s;m)$ satisfy the Eqs. (2) and (3), then using the Hamiltonian system (1) on the level set $\hat{\Gamma} = 0$, the components of $\gamma'(0;m)$ satisfy

$$u_3'(0;m) = -u_1'(0;m), \qquad u_4'(0;m) = u_2'(0;m), \qquad v_3'(0;m) = v_1'(0;m), \qquad v_4'(0;m) = -v_2'(0;m).$$

It is easily recognized that the vector $\gamma'(0;m)$ belongs to $\ker(Q+I)$. Now set

$$a=u_1'(0;m),\qquad b=u_2'(0;m),\qquad c=v_1'(0;m),\qquad d=v_2'(0;m),\qquad e=\|\gamma'(0;m)\|$$

and define Y_0 by

$$Y_0=\frac{1}{e}\begin{bmatrix} c & d & a & b & a & -b & -c & d\\ d & -c & b & -a & b & a & -d & -c\\ c & d & a & b & -a & b & c & -d\\ -d & c & -b & a & b & a & -d & -c\\ -a & b & c & -d & c & d & a & b\\ -b & -a & d & c & d & -c & b & -a\\ a & -b & -c & d & c & d & a & b\\ -b & -a & d & c & -d & c & -b & a \end{bmatrix}. \tag{7}$$

Let $\mathrm{col}_i(Y_0)$ denote the ith column of Y_0. Notice that $\mathrm{col}_5(Y_0)=\gamma'(0;m)/\|\gamma'(0;m)\|$. The last four columns of Y_0 form an orthonormal basis for $\ker(Q+I)$, while the first four columns of Y_0 form an orthonormal basis for $\ker(Q-I)$. Since Q is symmetric, its two eigenspaces are orthogonal, and so Y_0 is orthogonal. Note that $J\mathrm{col}_{4+i}(Y_0)=\mathrm{col}_i(Y_0)$ for $i=1,2,3,4$; in other words, multiplication by J maps $\ker(Q-I)$ bijectively to $\ker(Q+I)$. For P_1 the lower right 4×4 submatrix of Y_0 and P_2 the upper right 4×4 submatrix of Y_0, we have

$$Y_0=\left(J\begin{bmatrix}P_2\\P_1\end{bmatrix},\begin{bmatrix}P_2\\P_1\end{bmatrix}\right)=\begin{bmatrix}P_1 & P_2\\ -P_2 & P_1\end{bmatrix},$$

where $P_1^TP_1+P_2^TP_2=I$ and $P_1^TP_2=0$. This implies that Y_0 is symplectic. □

3.3. The existence of K

By Lemma 3.1 we have $Y_0^{-1}Y(2\pi)=W^4$ where $W=\Lambda D$ with $\Lambda=Y_0^{-1}QY_0$ and $D=B^{-1}S_GB$ for $B=Y(\pi/4)$. By Lemma 3.2, there exists an orthogonal and symplectic Y_0 such that Eq. (6) holds. Choose Y_0 as given in Eq. (7). The matrix $W=\Lambda D$ is then symplectic, i.e., $W^TJW=J$, because Λ is symplectic with multiplier -1, $\Lambda^TJ\Lambda=-J$, and S_G is symplectic with multiplier -1, $S_G^TJS_G=-J$, and B is symplectic.

Lemma 3.3. *With the given choice of Y_0, there exists a matrix K uniquely determined by $B=Y(\pi/4)$ such that*

$$\frac{1}{2}\left(W+W^{-1}\right)=\begin{bmatrix}K^T & 0\\ 0 & K\end{bmatrix}.$$

Proof. Since $W=\Lambda D$ where Λ and D are involutions, it follows that

$$W^{-1}=D\Lambda.$$

By the choice of Y_0, the form of the matrix Λ is given in Eq. (6). If we partition the symplectic matrix B into the four 4×4 submatrices,

$$B=\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}, \tag{8}$$

then the form of the inverse of B is

$$B^{-1}=\begin{bmatrix}A_4^T & -A_2^T\\ -A_3^T & A_1^T\end{bmatrix}.$$

Set

$$H=\begin{bmatrix}-G & 0\\ 0 & G\end{bmatrix}.$$

Then we have that

$$D=B^{-1}S_GB=\begin{bmatrix}K^T & L_1\\ -L_2 & -K\end{bmatrix}$$

where $K=A_3^THA_2+A_1^THA_4$, $L_1=A_4^THA_2+A_2^THA_4$, and $L_2=A_3^THA_1+A_1^THA_3$. It follows that K is uniquely determined by B, that

$$W=\Lambda D=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix}K^T & L_1\\ L_2 & -K\end{bmatrix}=\begin{bmatrix}K^T & L_1\\ L_2 & K\end{bmatrix}, \tag{9}$$

and that

$$W^{-1} = DA = \begin{bmatrix} K^T & L_1 \\ L_2 & -K \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} K^T & -L_1 \\ -L_2 & K \end{bmatrix}.$$

Thus

$$\frac{1}{2}\left(W + W^{-1}\right) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix} \tag{10}$$

for a K uniquely determined by $B = Y(\pi/4)$ as was desired. □

It has been shown [2] that the symplectic matrix W is spectrally stable, i.e., all of its eigenvalues have modulus 1, if and only if all of the eigenvalues of K are real and have absolute value smaller than or equal to 1. The particular relationship between the eigenvalues of W and K given tacitly in Lemma 3.3 is as follows. The map $f : \mathbb{C} \to \mathbb{C}$ given by $f(\lambda) = (1/2)(\lambda + 1/\lambda)$ takes an eigenvalue of W to an eigenvalue of $(1/2)(W + W^{-1})$. Note that the map f satisfies $f(\lambda) = f(1/\lambda)$. For an eigenvalue λ of W, the eigenvalue $f(\lambda)$ of $(1/2)(W + W^{-1})$ is an eigenvalue of K. If λ is an eigenvalue of the symplectic matrix W, then $1/\lambda$, $\bar{\lambda}$, and $1/\bar{\lambda}$ are also eigenvalues of W. When λ has modulus one, then $\lambda = 1/\bar{\lambda}$ and $1/\lambda = \bar{\lambda}$, and so $f(\lambda) = f(\bar{\lambda})$ which is a real number with absolute value smaller than or equal to 1. Thus a complex conjugate pair of eigenvalues of W of modulus one corresponds to a real eigenvalue of K with absolute value smaller than or equal to 1. When λ is real, it is nonzero because W is symplectic, and $f(\lambda) = f(1/\lambda)$ which is a real number with absolute value greater than 1. Thus a reciprocal pair λ and $1/\lambda$ of real nonzero eigenvalues of W corresponds to a real eigenvalue of K with absolute value greater than 1. When λ is not real and has a modulus other than 1, then $f(\lambda) = f(1/\lambda)$ and $f(\bar{\lambda}) = f(1/\bar{\lambda})$, with $f(\lambda)$ and $f(\bar{\lambda})$ as complex conjugate eigenvalues of K with nonzero imaginary part. Thus, the four eigenvalues λ, $1/\lambda$, $\bar{\lambda}$, and $1/\bar{\lambda}$ of W correspond to a complex conjugate pair of eigenvalues of K with nonzero imaginary part.

3.4. The form of K

We will show that one of the eigenvalues of K is 1, and the remaining three eigenvalues of K are determined by the lower right 3×3 submatrix of K. Let c_i denote the ith column of $B = Y(\pi/4)$.

Lemma 3.4. *With the given choice of Y_0, the matrix K uniquely determined by $B = Y(\pi/4)$ is*

$$\begin{bmatrix} 1 & * & * & * \\ 0 & c_2^T S_G J c_6 & c_2^T S_G J c_7 & c_2^T S_G J c_8 \\ 0 & c_3^T S_G J c_6 & c_3^T S_G J c_7 & c_3^T S_G J c_8 \\ 0 & c_4^T S_G J c_6 & c_4^T S_G J c_7 & c_4^T S_G J c_8 \end{bmatrix}.$$

Proof. We begin by showing that 1 is an eigenvalue of W by identifying a corresponding eigenvector. Since $Y(\pi/2) = S_G Y_0 B^{-1} S_G B$ (Eq. (5)) and $Q = S_F^T S_G$, it follows that

$$\begin{aligned} W &= Y_0^{-1} Q Y_0 B^{-1} S_G B \\ &= Y_0^{-1} S_F^T S_G Y_0 B^{-1} S_G B \\ &= Y_0^{-1} S_F^T Y(\pi/2). \end{aligned}$$

Set

$$v = Y_0^{-1} \gamma'(0; m).$$

The orthogonality of Y_0 and $\mathrm{col}_5(Y_0) = \gamma'(0; m)/\|\gamma'(0; m)\|$ imply that

$$v = Y_0^T \gamma'(0; m) = \|\gamma'(0; m)\| e_5,$$

where $e_5 = [0, 0, 0, 0, 1, 0, 0, 0]^T$. Since $Y(s)$ is a fundamental matrix, then $\gamma'(s; m) = Y(s) Y_0^{-1} \gamma'(0; m)$. Hence,

$$\begin{aligned} Wv &= Y_0^{-1} S_F^T Y(\pi/2) v \\ &= Y_0^{-1} S_F^T Y(\pi/2) Y_0^{-1} \gamma'(0; m) \\ &= Y_0^{-1} S_F^T \gamma'(\pi/2; m). \end{aligned}$$

Since $S_F \gamma(s; m) = \gamma(s + \pi/2; m)$ and $S_F^{-1} = S_F^T$, we have that

$$\gamma'(s; m) = S_F^{-1} \gamma'(s + \pi/2; m) = S_F^T \gamma'(s + \pi/2; m).$$

Setting $s = 0$ in this gives

$$\gamma'(0; m) = S_F^T \gamma'(\pi/2; m).$$

From this it follows that

$$\begin{aligned} Wv &= Y_0^{-1}S_F^T\gamma'(\pi/2;m) \\ &= Y_0^{-1}\gamma'(0;m) \\ &= v. \end{aligned}$$

Thus 1 is an eigenvalue of W and $v = \|\gamma'(0;m)\|e_5$ is a corresponding eigenvector.

Next, we show that the first column of K is $[1,0,0,0]^T$. Since $Wv = v$, then $We_5 = e_5$. From the form of W given in Eq. (9), it follows that

$$e_5 = We_5 = \begin{bmatrix} L_1[1,0,0,0]^T \\ K[1,0,0,0]^T \end{bmatrix}.$$

This implies that

$$K\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

from which it follows that the first column of K is $[1,0,0,0]^T$.

Finally we show that the lower right 3×3 submatrix of K has the prescribed entries. Since Y_0 is symplectic, the matrix $B = Y(\pi/4)$ is symplectic. Hence B satisfies $J = B^TJB$, and so

$$B^{-1} = -JB^TJ.$$

For $W = \Lambda D$ with $D = B^{-1}S_GB$ where S_G satisfies $S_GJ = -JS_G$ we then obtain

$$\begin{aligned} W &= \Lambda B^{-1}S_GB \\ &= \Lambda(-JB^TJ)S_GB \\ &= -\Lambda JB^TJS_GB \\ &= -\Lambda JB^T(-S_GJ)B \\ &= \Lambda JB^TS_GJB. \end{aligned}$$

Writing B in the block partition form given in Eq. (8), it follows that

$$\Lambda JB^T = \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}B^T = \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}A_1^T & A_3^T\\ A_2^T & A_4^T\end{bmatrix} = \begin{bmatrix}A_2^T & A_4^T\\ A_1^T & A_3^T\end{bmatrix}. \tag{11}$$

Let $\mathrm{col}_i(S_GJB)$ denote the ith column of S_GJB. Then $\mathrm{col}_i(S_GJB) = S_GJc_i$ where c_i is the ith column of $B = Y(\pi/4)$. This and Eq. (11) imply that the (i,j) entry of W is then $c_i^TS_GJc_j$. But Eq. (9) implies that the $(6,6)$ entry of W is the $(2,2)$ entry of K. Continuing in this manner we find the remaining entries of the lower right 3×3 submatrix of K to be given as prescribed. □

3.5. A stability theorem

The characteristic multipliers of $\gamma(s;m)$ are the eigenvalues of W^4 which are the fourth powers of the eigenvalues of W. As was shown in the proof of Lemma 3.4, an eigenvalue of K is 1. Because of Eq. (10), an eigenvalue of W is 1 with algebraic multiplicity (at least) 2. This accounts for two of the four known eigenvalues of 1 for W^4. Our numerical calculations show that -1 is an eigenvalue of K and hence of W for all $0 < m \le 1$. This accounts for the remaining two known eigenvalues of 1 for W^4.

When W is spectrally stable, the eigenvalues of K are the real parts of the eigenvalues of W. If 0 is an eigenvalue of K, then $\pm i$ are eigenvalues of W and so the algebraic multiplicity of 1 as an eigenvalue of W^4 is at least 6. If $1/\sqrt{2}$ is an eigenvalue of K, then $1/\sqrt{2} \pm i/\sqrt{2}$ are eigenvalues of W, and if $-1/\sqrt{2}$ is and eigenvalue of K, then $-1/\sqrt{2} \pm i/\sqrt{2}$ are eigenvalues of W; both these imply that -1 is a repeated eigenvalue of W^4. So when the remaining two eigenvalues λ_1 and λ_2 of K are real, distinct, have absolute value strictly smaller than one, and none of them are equal to 0 or $\pm 1/\sqrt{2}$, then the symmetric periodic SBC orbit is linearly stable, i.e., W, and hence W^4, is spectrally stable as well as semisimple when restricted to the four dimensional W-invariant subspace of $\mathbb{R}^8$ determined by the two distinct modulus one complex conjugate pairs of eigenvalues of W. On the other hand, if one of λ_1 or λ_2 is real with absolute value bigger than 1, or is complex with a nonzero imaginary part, then the symmetric periodic SBC orbit is not spectrally stable, but is linearly unstable. The proof of the following result about the linear stability type for the symmetric periodic SBC orbits in the PPS4BP follows from all of the lemmas and subsequent comments presented in this section.

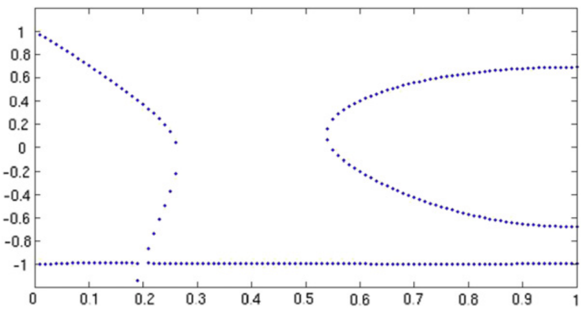

Fig. 1. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$, when real, of the 3×3 lower right submatrix of K over $0 < m \leq 1$.

Theorem 3.5. *The symmetric periodic SBC orbit $\gamma(s; m)$ of period $T = 2\pi$ and energy $\hat{E}(m)$ is spectrally stable in the* PPS4BP *if and only if λ_1 and λ_2 are real and have absolute value smaller or equal to* 1. *If λ_1 and λ_2 are real, distinct, have absolute value strictly smaller than* 1, *and none of them are equal to* 0 *or* $\pm 1/\sqrt{2}$, *then $\gamma(s; m)$ is linearly stable in the* PPS4BP.

4. Numerical results

We computed $Y(\pi/4)$ using our trigonometric polynomial approximations of $\gamma(s; m)$ for each m starting at $m = 1$ and decreasing by 0.01 until we reached $m = 0.01$, and the Runge–Kutta order 4 algorithm coded in MATLAB, with a fixed time step of

$$\frac{\pi/4}{50000} = \frac{\pi}{200000}.$$

From the needed columns of $Y(\pi/4)$, we computed the entries of the lower right 3×3 submatrix of K as given in Lemma 3.4, and then computed the eigenvalues λ_1, λ_2, and λ_3 of this 3×3 matrix. We have plotted these three eigenvalues, when real, as functions of m in Fig. 1. One of these eigenvalues is real and stays close to -1 for all $m \in (0, 1]$ except at $m = 0.20$; label this eigenvalue λ_3.

The remaining two eigenvalues of K determine the linear stability type of $\gamma(s; m)$. Label these λ_1 and λ_2 where the value of λ_1 at $m = 0.01$ is 0.9743145796 (see Fig. 1), and the value of λ_2 at $m = 0.01$ is -50.70044516 (not shown in Fig. 1). As m approaches 0 from above, it appears in Fig. 1 that λ_1 approaches 1 from below, while λ_2 stays smaller than -50.70044516. As m increases from 0.01, the value of λ_1 decreases, crossing $1/\sqrt{2}$ for a value m_1 in (0.09, 0.10), and crossing 0 at a value m_4 in (0.26, 0.27), while λ_2 increases to the value -1.146019443 at $m = 0.19$, momentarily disappearing in Fig. 1 along with λ_3 at $m = 0.20$, reappearing at $m = 0.21$ with a value of -0.8641436215, continuing to increase, crossing $-1/\sqrt{2}$ for a value m_3 in (0.22, 0.23), until at a value m_5 in $(m_4, 0.27)$, we have $\lambda_1 = \lambda_2 < 0$. As m increases from m_5, the eigenvalues λ_1 and λ_2 form a complex conjugate pair with nonzero imaginary part, and thus do not appear in Fig. 1. At a value m_6 in (0.53, 0.54), we have λ_1 and λ_2 reappearing in Fig. 1, with $\lambda_1 = \lambda_2 > 0$. As m increases from m_6, the value of λ_1 increases and the value of λ_2 decreases, with λ_2 crossing 0 for a value m_7 in (0.54, 0.55), and with the values of λ_1 and λ_2 at $m = 1$ being respectively,

$$0.6941364299, -0.6802222699, \tag{12}$$

where the first of these is slightly smaller than $1/\sqrt{2}$, and the latter is slighter larger than $-1/\sqrt{2}$. Except when $m = m_1$, these changes in the values of λ_1 and λ_2 account for the changes in the linear stability type of $\gamma(s; m)$ as m varies over (0, 1]. There is no change in the linear stability type of $\gamma(s; m)$ as m crosses m_1 because $\lambda_2 < -1$ at m_1. However we have that -1 is a repeated characteristic multiplier of $\gamma(s; m_1)$.

4.1. Estimates of the critical values of m

In addition to the critical values of m already identified, there is another critical value m_2 near 0.20. At $m = 0.20$ the eigenvalues λ_2 and λ_3 disappear in Fig. 1 because they form the complex conjugate pair with a small nonzero imaginary part,

$$-0.9972588720 \pm 0.008650400165i.$$

To better understand what is happening with λ_2 for m close to 0.20 we numerically computed $Y(\pi/4)$ for m between 0.19 and 0.21 at increments of 0.001, and then computed λ_2 and λ_3. For $m = 0.190, 0.191, \ldots, 0.199$ we have $\lambda_2 < -1$ with

$$\lambda_2 = -1.008505655$$

at $m = 0.199$, while for $m = 0.202, 0.203, \ldots, 0.209$ we have $\lambda_2 > -1$ with

$$\lambda_2 = -0.9772249606 \tag{13}$$

at $m = 0.202$. For $m = 0.201$, the eigenvalues λ_2 and λ_3 form a complex conjugate pair with a small nonzero imaginary part,

$$-0.9902780905 \pm 0.008216343352i.$$

The small imaginary parts of the values of λ_2 and λ_3 for $m = 0.200$ and $m = 0.201$, and the monotonicity of λ_2 as m increases, suggest that there is a unique value m_2 in the interval $(0.199, 0.202)$ at which $\lambda_2 = \lambda_3 = -1$. By linear interpolation using the values of λ_2 at $m = 0.199$ and $m = 0.202$, we get the estimate

$$m_2 \approx 0.1998157417.$$

To get better estimates of the critical values m_3, m_4, m_5, m_6, and m_7 we numerically computed $Y(\pi/4)$ for values of m in 0.001 increments near m_i, $i = 3, 4, 5, 6, 7$, and then computed the values of λ_1 and λ_2. At m_3 in $(0.22, 0.23)$ we have $\lambda_2 = -1/\sqrt{2}$. For $m = 0.220, 0.221, 0.222$ we have $\lambda_2 < -1/\sqrt{2}$ with

$$\lambda_2 = -0.712389793$$

at $m = 0.222$, and for $m = 0.223, 0.223, \ldots, 0.230$ we have $\lambda_2 > -1/\sqrt{2}$ with

$$\lambda_2 = -0.7000674602$$

at $m = 0.223$. Linear interpolation of the values of λ_2 at $m = 0.222$ and $m = 0.223$ gives the estimate

$$m_3 \approx 0.2224287348.$$

At m_4 in $(0.26, 0.27)$ we have $\lambda_1 = 0$. For $m = 0.260, 0.261, 0.262$ we have $\lambda_1 > 0$ with

$$\lambda_1 = 0.01436928394$$

at $m = 0.262$, while for $m = 0.263, 0.264$ we have $\lambda_1 < 0$ with

$$\lambda_1 = -0.007229915575$$

at $m = 0.263$. Linear interpolation of the values of λ_1 at $m = 0.262$ and $m = 0.263$ gives the estimate

$$m_4 \approx 0.2626652693.$$

At m_5 in $(m_4, 0.27)$ we have $\lambda_1 = \lambda_2 < 0$. For $m = 0.263, 0.264$ we have $\lambda_1 < 0$ and $\lambda_2 < 0$, while for $m = 0.265, 0.266, \ldots, 0.270$ we have that λ_1 and λ_2 form a complex conjugate pair with a nonzero imaginary part that is increasing in absolute value as m increases. Thus

$$m_5 \in (0.264, 0.265).$$

At m_6 in $(0.53, 0.54)$ we have $\lambda_1 = \lambda_2 > 0$. For $m = 0.530, 0.531, \ldots, 0.538$, we have that λ_1 and λ_2 form a complex conjugate pair with a nonzero imaginary part that is decreasing in absolute value as m increases, and for $m = 0.539, 0.540$ we have that $\lambda_1 > 0$ and $\lambda_2 > 0$. Thus

$$m_6 \in (0.538, 0.539).$$

At m_7 in $(0.54, 0.55)$ we have $\lambda_2 = 0$. For $m = 0.540, 0.541, \ldots, 0.546$ we have that $\lambda_2 > 0$ with

$$\lambda_2 = 0.0006982328764$$

at $m = 0.546$, while for $m = 0.547, 0.548, 0.549, 0.550$ we have $\lambda_2 < 0$ with

$$\lambda_2 = -0.005921393183$$

at $m = 0.547$. Linear interpolation of the values of λ_2 at $m = 0.546$ and $m = 0.547$ gives the estimate

$$m_7 \approx 0.5461054792.$$

From the above numerical results and Theorem 3.5, we conclude the following about the linear stability type of the periodic orbit $\gamma(s; m)$ for $m \in (0, 1]$. We have linearly stable when m belongs to

$$(m_2, m_3) \cup (m_3, m_4) \cup (m_4, m_5) \cup (m_6, m_7) \cup (m_7, 1],$$

at least spectral stability when m is one of $m_2, m_3, m_4, m_5, m_6, m_7$, and linear instability when m belongs to

$$(0, m_2) \cup (m_5, m_6).$$

4.2. Interpretation of numerical results

We have confirmed numerically that the equal mass symmetric periodic SBC orbit $\gamma(s; 1)$ is linearly stable in the PPS4BP. Since the values of λ_1 and λ_2 listed in (12) are the real parts of fourth roots of the modulus one nontrivial characteristic multipliers of $\gamma(s; m)$, the nontrivial characteristic multipliers of $\gamma(s; 1)$ are the two distinct complex conjugate pairs of modulus one,

$$-0.9888710746 \pm 0.1487749902i,$$
$$-0.9973574665 \pm 0.07265042297i.$$

These, which are near -1, agree numerically with the eigenvalues of the monodromy matrix for $\gamma(s; 1)$ we obtained in [4].

We first describe the motion of nontrivial characteristic multipliers of the linearly or spectrally stable $\gamma(s; m)$ for m varying in $[m_6, 1]$. As m decreases from 1 towards m_7, the nontrivial characteristic multipliers of $\gamma(s; m)$ are two distinct complex conjugate pairs of modulus one that, starting near -1 when $m = 1$, move away from -1 towards 1 on the unit circle. As m approaches m_7 from above, the complex conjugate pair closest to 1 coalesces into a repeated 1, and then as m decreases below m_7 towards m_6, it moves away from 1 as a complex conjugate pair of modulus one. As m approaches m_6 from m_7, the one conjugate pair of modulus one moving towards 1 coalesces with the complex conjugate pair of modulus one moving away from 1 to form a double complex conjugate pair of modulus one at m_6.

An estimate of the value of this double complex conjugate pair when $m = m_6$ is obtained from the nontrivial characteristic multipliers of the linearly stable $\gamma(s; 0.539)$. Since

$$\lambda_1 = 0.1425261155, \qquad \lambda_2 = 0.08595095311$$

when $m = 0.539$, the nontrivial characteristic multipliers of $\gamma(s; 0.539)$ are the two complex conjugate pairs of modulus one,

$$0.8407916212 \pm 0.5413588917i,$$
$$0.9413360780 \pm 0.3374705738i.$$

As m decreases from $m = 0.539$ to m_6, the first pair of these moves toward 1 and the second pair moves away from 1. At m_5, they coalesce into a double complex conjugate pair of modulus one whose values are estimated by the midpoints along the unit circle of the nontrivial characteristic multipliers of $\gamma(s; 0.539)$, i.e., the point on the unit circle determined by the midpoint of the angles of the nontrivial characteristic multipliers in the first quadrant, and its complex conjugate in the second quadrant. This estimate is

$$0.8968764662 \pm 0.4422811373i.$$

We next describe the motion of the nontrivial characteristic multipliers of $\gamma(s; m)$ as m varies in $[m_2, m_5]$. The characteristic multipliers of $\gamma(s; m_5)$ are a double complex conjugate pair, an estimate of which is given by the midpoints along the unit circle of the characteristic multipliers of $\gamma(s; 0.264)$. When $m = 0.264$, we have

$$\lambda_1 = -0.04192545394, \qquad \lambda_2 = -0.1129277667,$$

from which it follows that the nontrivial characteristic multipliers of $\gamma(s; 0.264)$ are the two complex conjugate pairs of modulus one,

$$0.9859627678 \pm 0.1669653273i$$
$$0.8992796029 \pm 0.4373742057i.$$

The midpoints along the unit circle of these are

$$0.9522683127 \pm 0.3052622816i.$$

As m decreases from m_5 towards m_4, the complex conjugate pair of modulus one closest to 1 approaches 1 on the unit circle, and the complex conjugate pair of modulus one farthest from 1 moves away from 1. As m approaches m_4, the complex conjugate pair closest to 1 coalesces into a repeated 1, and then as m decreases below m_4, it moves away from 1 as a complex conjugate pair of modulus one. As m decreases from m_4 towards m_3, the complex conjugate pair of modulus one farthest away from 1 continues to move towards -1 while the complex conjugate pair near 1 continues to move away from 1. As m approaches m_3, the complex conjugate pair of modulus one farthest from 1 coalesces into a repeated -1, and then as m decreases below m_3, it moves away from -1 as a complex conjugate pair of modulus one. As m decreases away from m_3 towards m_2, the complex conjugate pair of modulus one near -1 moves towards 1, while the conjugate pair of modulus one near 1 continues to move away from 1. At $m = 0.202$, which is in (m_2, m_3), we have $\lambda_1 = 0.3653691747$ and the value of λ_2 is given in (13). The nontrivial characteristic multipliers of $\gamma(s; 0.202)$ are the two complex conjugate pairs of modulus one,

$$0.07460946747 \pm 0.9972128295i$$
$$0.6559715877 \pm 0.7547855829i.$$

As m approaches m_2 from above, the complex conjugate pair that is moving away from -1 approaches 1 to form a double 1 at $m = m_2$, while the complex conjugate pair of modulus one that is moving away from 1 continues to move away from 1.

Within (m_2, m_3) there is a value m^* at which the nontrivial characteristic multipliers of $\gamma(s; m)$ coalesce into a double complex conjugate pair of modulus one as m approaches m^*. The value of m^* lies in $(0.204, 0.205)$. This is because at $m = 0.204$ we have

$$\lambda_1 = 0.3578570178, \qquad \lambda_2 = -0.9465699740$$

from which the nontrivial characteristic multipliers of $\gamma(s; 0.204)$ are

$$0.1067051182 \pm 0.9942907109i$$
$$0.2544945187 \pm 0.7199386086i,$$

each of modulus one, and at $m = 0.205$ we have

$$\lambda_1 = 0.3540716279, \qquad \lambda_2 = -0.9325602084$$

from which the nontrivial characteristic multipliers of $\gamma(s; 0.205)$ are

$$0.1228007698 \pm 0.9924313432$$
$$0.09323864917 \pm 0.9956437889,$$

each of modulus one. Since $m^* \in (0.204, 0.205)$, the values of λ_1 and λ_2 at m^* satisfy

$$0.3540716279 < \lambda_1 < 0.3578570178, \qquad -0.9465699740 < \lambda_2 < -0.9325602084.$$

The double complex conjugate pair of modulus one nontrivial characteristic multipliers at $m = m^*$ might indicate that $\gamma(s; m^*)$ is at best spectrally stable, but because the corresponding values of λ_1 and λ_2 when $m = m^*$ are real, distinct, with absolute value strictly smaller than 1, and neither is equal to 0 or $\pm 1/\sqrt{2}$, we have by Theorem 3.5 that $\gamma(s; m^*)$ is linearly stable.

5. Work in progress

We are currently investigating the analytic existence and numerical linear stability of another mass-parameterized family of symmetric periodic orbits with regularizable collisions in the PPS4BP. These periodic orbits have alternating binary collisions of the symmetric pairs of equal masses. The analytic existence of these periodic orbits in the rhomboidal four-body 1, m, 1, m problem has been given by Yan [21] for $m = 1$, and by Shibayama [10] for arbitrary $m > 0$. For $m = 1$, the linear stability of this periodic orbit in the rhomboidal four-body problem has been numerically established by Yan [21] using Roberts' symmetry reduction method. We are extending the analytic existence of these periodic orbits from the rhombodial four-body problem to the PPS4BP, similar to what we did in [4]. We are also extending the linear stability analysis to arbitrary $0 < m < 1$ in the rhomboidal four-body problem and in the PPS4BP to find those intervals of values of m for which these periodic orbits are linearly stable in the respective problems.

Acknowledgments

The authors express thanks for the referee's valuable feedback that improved the presentation and results of this paper.

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